

TASK SUPPORT APPENDIX FOR DEVELOPING THINKING IN ALGEBRA

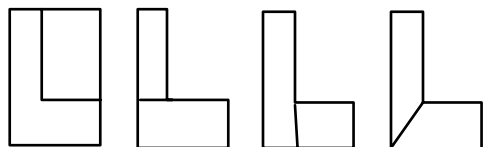
Task 1.2.3a Finding Areas

Find at least three different ways to decompose this shape into rectangles to find its area. You have to decide on appropriate lengths for yourself, but the aim is to express a method which works in general, for all 'L-shapes'.



Suggestion

Here are four ways learners have come up with. In the first, the area of the L-shape is the difference between the areas of two rectangles.

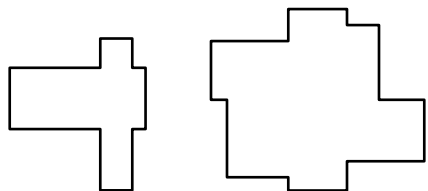


Task 1.2.3b Finding Complicated Areas

Make up a figure from rectangles which you think make it difficult to see how to do it with a minimum number of rectangles.

Suggestion

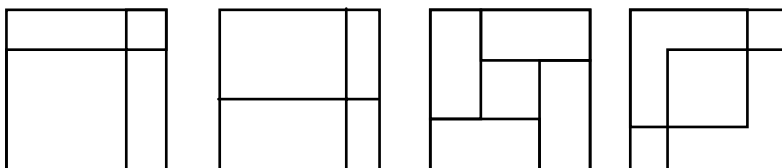
Gluing rectangles together produces a shape, but what determines how few rectangles might be required for a given shape? And what is the minimum information that has to be supplied to find the area of the shape? What numbers can occur as the number of right-angles in a shape made up from rectangles?



The first was drawn with two rectangles overlapping, but requires three to work out the area. The second was drawn with five rectangles.

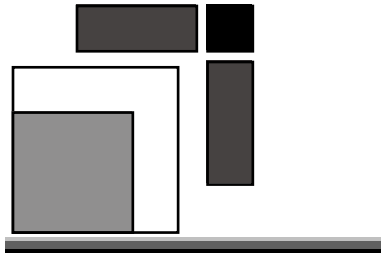
Task 1.2.4 Reading Areas

Interpret each of the following diagrams as general statements about areas in as many ways as possible.

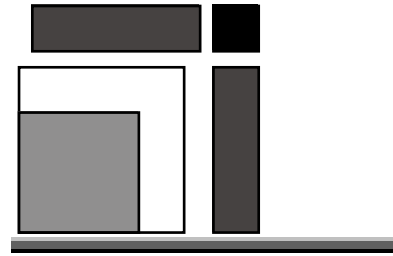


Suggestion

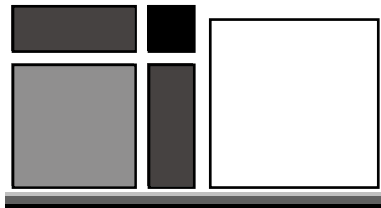
The following diagrams are intended to suggest some readings for the first diagram



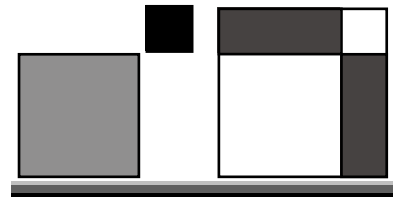
On the left, the area of the whole white square take away the area of the light-shaded square is the same area as the area of the black square plus twice the area of a dark-shaded rectangle.



On the right, the area of the whole white square take away the area of the light-shaded square is twice the area of a dark-shaded rectangle take away the area of the black square



On the left, the area of the white square is the sum of the areas of the shaded squares and rectangles.



On the right, the area of the white square take away twice the area of a shaded rectangle is the sum of the areas of the two squares.

There are many more arrangements. If the sides of the light-shaded and black squares are denoted by a and b respectively, then each of these ways of seeing translate into symbolic expressions:

$$(a + b)^2 - a^2 = 2ab + b^2; (a + b)^2 - a^2 = 2(a + b)b - b^2; (a + b)^2 = a^2 + 2ab + b^2; (a + b)^2 - 2ab = a^2 + b^2$$

If the side of the white square is denoted by c and the side of the light-shaded square by a , then

$$c^2 - a^2 = 2a(c - a) + (c - a)^2; c^2 - a^2 = 2c(c - a) + (c - a)^2; \text{ and}$$

$$c^2 = a^2 + 2a(c - a) + (c - a)^2; c^2 - 2a(c - a) = a^2 + (c - a)^2$$

Checking that the symbols accurately express the various regions requires careful discernment!

The third diagram in the task can be read as four times the area of the rectangle plus the area of the small square is the area of the large square: $4ab + (b - a)^2 = (a + b)^2$. For areas, a and b have to be non-negative, at least for the time being.

For the last diagram, let a be the length of side of the largest square which is not the whole square, and let b be the side of the smallest square. The diagram can then be read as twice the area of the largest square which is not the whole square, plus twice the area of the smallest square equals the area of the middle square plus the area of the whole square: $2a^2 + 2b^2 = (a - b)^2 + (a + b)^2$. Another reading is that twice the difference of the areas of two squares (the largest not the whole and the middle one) plus the area of the middle square is the area of the whole square minus twice the area of the small square: $2(a^2 - (a - b)^2) + (a - b)^2 = (a + b)^2 - 2b^2$. There are many more possible readings!

Task 1.3.2c VAT & Discount (continued)

Suppose a warehouse offers a discount of $D\%$ and furthermore, no VAT to be paid by the customer. What is the effective discount being offered?

Suggestion

Use a particular case to work out what computations need to be done, then generalise from that.

If the initial price is $\text{£}P$, then the total price should be $(100 + \text{VAT}\%)$ of P . The price charged is $(100 - D\%)$ of P , which is $(100 - D\%)/(100 + \text{VAT}\%)$ of $(100 + \text{VAT}\%)$ of P . What is the equivalent actual discount? It is $100 - (100 - D\%)/(100 + \text{VAT}\%)$, since that is the amount to be subtracted from 100 which when multiplied by the actual price gives the price being charged.

Task 1.3.4 Customs & Excise

How does a shop work out how much VAT they owe Customs & Excise if all their shop prices are quoted as inclusive of VAT?

Suggestion

Taking the suggestion in the comment, try an item priced originally at £100 before VAT. Then the quoted price will be £100 + 17.5% of £100, or denoting VAT by v as a decimal, $£100(1 + v)$. More generally, if the pre-VAT price is P , then the VAT inclusive price will be $P(1 + v)$. To recover the pre-VAT price from this, you need to divide by $(1 + v)$. The Customs and Excise get the extra, which is $Pv/(1 + v)$. The shop needs to 'know' the multiplier $v/(1 + v)$, then multiply this by its total sales to get the total VAT bill.

Task 2.1.1 Adding Constraints

Write down a number which, when you subtract 1, the result is divisible by 2. Express all such numbers.

Write down a number which, when you subtract 1, the result is divisible by 2 and by 3. Express all such numbers.

Write down a number which, when you subtract 1, the result is divisible by 2 and by 3, and by 4. Express all such numbers.

Generalise!

Suggestion

The quickies in chapter 1 were similar to the first part, just expressed as 'one more than' rather than 'if you subtract'. One approach to the second part is to write down all the numbers which meet the first constraint and then pick out the ones required. You can make a list which displays the pattern (1, 3, 5, 7, ...), or you can express them all in general terms $(1 + 2n)$. From the list you can pick out 7, 13, 19, and then see a pattern. From the general form you can re-express the same number as $1 + 3m$ and since the two have to be equally, $2n = 3m$. This means that n is a multiple of 3, and m is a multiple of 2, so multiples of 6 are required.

Learning from experience, a reasonable conjecture for the third part would be $1 + 2 \times 3 \times 6 \times n$. proceeding from the list approach, 1, 7, 13, 19, 25, 31, 37, ... 13 and 25 both work, so in fact the general form is $1 + 12n$. The smallest number divisible by each of 2, 3, and 4 is 12 not 24. It is known as the *least common multiple* of 2, 3, and 4.

Task 2.1.3 Odd Sum

Someone announced that the sum of consecutive odd numbers is always the difference of two squares. What do you think?

Suggestion

Did you work systematically? Some people start from 1 (e.g. 1, $1 + 3$, $1 + 3 + 5$, ...)? There may be some surprise that those sums are all square numbers! A conjecture suggests itself: perhaps the general assertion is weaker than need be? Then, starting at 3 (e.g. 3, $3 + 5$, $3 + 5 + 7$, ...), the sums turn out to be 1 less than a perfect square. Quickly moving on (there can be a sense of urgency to find out ...) to starting with 5 (e.g. 5, $5 + 7$, $5 + 7 + 9$, ...) the sums turn out to be 4 less than a perfect square. Soon a general conjecture can be expressed!

Some people are systematic in a different way by looking at pairs first: $1 + 3 = 4 = 2^2 - 0^2$, $3 + 5 = 8 = 3^2 - 1^2$, $5 + 7 = 12 = 4^2 - 2^2$, and generalising. Then they turn to triples, and so on.

The aim is not to do a lot of calculations but to try to predict the two square numbers whose difference is the same as the sum. The sequences starting at 1 may not be very helpful in this regard, though it may be

possible to spot which number needs to be squared! It is worth expressing this as a generality (its status is as a conjecture for you don't know if it is always true) before going on the sequences starting from 3, and perhaps from some other odd number like 7 or 11.

A moment's pause from calculations might produce the idea that any sequence of consecutive odd numbers is formed by taking two such sequences both starting at 1 and subtracting the shorter from the longer. Now the work on expressing the sum of consecutive odd numbers starting at 1 as a perfect square provides just the information you need to express any sum of consecutive odd numbers as a difference of two squares.

Task 2.4.2a Number-Line Movements

Imagine a number-line. Imagine a copy of it sitting right on top. Now imagine that the copy is rotated through 180° about the point marked 0. Where on the original number-line would you find the 3 on the copy? In other words, where has 3 moved to? What about 5? 7? Generalise!

Suggestion

It may help to draw a number line but then simply look at it without touching it or adding anything to it as you follow the instructions in this and similar tasks.

Task 3.3.1a Multiple Names

In how many different ways can you depict one-half?

Suggestion

Equivalent fractions (infinitely many different expressions, but perhaps all of one type). The multiples of numerator and denominator do not even have to be whole number multiples!

As a division of two numbers (rather similar to equivalent fractions, but possibly perceived differently). The numbers can be whole numbers or expressed as decimals, fractions, negative as well as positive. The division sign could be a horizontal bar, a sloping bar, or a division sign.

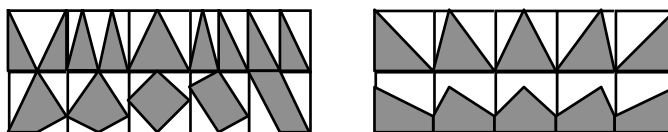
Decimal notation (two forms, one involving recurring nines).

Scientific notation and variants of it (multiply a number by a power of ten).

Bases other than ten could be used, such as base 2, or base 3.

One-half can be depicted in a myriad of ways using different aspects of one shape shaded in, or different aspects of two or more shapes shaded in, or some objects from a collection shaded in, etc..

There is a classic computer-animated film called 'Take Half' which suggests different ways of shading in one-half of a square, and there is a poster which shows some of the frames from the film. Here is a sample of the possibilities.



Task 3.3.1b Multiple Names

What is the same, and what is different about the following four expressions:

$$(x+1)(x-3) \quad (x-1)^2 - 4 \quad x^2 - 2x - 3 \quad x(x-2) - 3$$

Suggestion

Start to generalise by changing the numbers.

The first expression can be generalised to $(x + a)(x + b)$ by treating the original as using 1 and -3 , or as $(x + a)(x - b)$ by treating the original as using 3. The task then becomes one of working out what the other equivalent forms should be. If you know how to manipulate letters, then you 'do some algebra'; if not, you use some carefully chosen examples to try to detect a pattern (how does the -2 in the second expression relate to the 1 and the -3 in the first?)

Using the first form, the second expression will then be $(x + (a + b)/2)^2 + ab - (a + b)^2/4$, and the third expression will be $x^2 + (a + b)x + ab$

Task 3.3.3 Key Sequences

In the comment, the question arose as to whether one person could add before multiplying, and the other subtract after multiplying. The answer is no, if all numbers are positive, but yes if the numbers added and subtracted differ in sign, or both were zero!

Task 3.3.4b Multiple Expressions

Find a way to see structure in the house-pictures which correspond to the following expressions of generality for the number of matchsticks needed for the p^{th} picture:

$$2 + (p - 1) + (1 + 2p + 1); \quad 2 + (3p - 1) + 2; \quad 4 + 3(p - 1) + 2; \quad 3(p + 1)$$

Suggestion

For each term added together, look for some collection of sticks which could correspond to that term and that contributes to each and every picture in the sequence. For example, the last expression suggests finding three lots of $p + 1$ sticks arranged in some generic or generalisable arrangement (for example the left end wall and roof stick with the right end wall stick, and then triples of roof, top and bottom sticks).

Task 3.3.6 Painted Cube

A cube has been painted on all faces. It is then cut up into 27 cubelets by two plane cuts parallel to each face. How many cubelets are painted on how many faces?



Generalise!

Suggestion

The total number of cubelets in general is n^3 . The number painted on just three faces (three faces exposed) is always 8 (the corner cubelets). The number painted on just two faces (two faces exposed) is $12(n - 2)$ since there are $n - 2$ on each of 12 edges not including at the corners. The number painted on just one face is $6(n - 2)^2$ since there are 6 faces and there are $n - 2$ by $n - 2$ cubelets whose only exposed face is on that face of the cube. Finally, there are $(n - 2)^3$ cubelets inside that don't get painted at all. Some care is needed however, as this count may only work for $n \geq 2$. (Check it for $n = 1$ and explain the numbers).

Task 3.4.3b Number Tracking Extended

Preserving the arithmetic operation sequence, first an add, then a times, then a subtract, then a divide, make up some sequences which always give even answers. Then make up some sequences which always give integer answers. What do you notice about the 'divide number' and the others? Try to express a general form for sequences which always yield integer answers. Try changing the order of the four operations.

Suggestion

Either from experience of trying to construct examples, or from expressing the operations in general, it turns out that the 'divide number' must in fact divide the 'times number' and the 'subtract number' in order to produce an integer answer.

The operations sequence can be expressed in general as

$$((x + a) \times b - c) / d \text{ which simplifies to } (bx + ab - c) / d \text{ where } d \neq 0.$$

Since x can be any integer at all, the d will have to divide into b , which means it already divides into ab which forces it to also divide into c . So a general form for integer-answered operation sequences might be add a ; times by bd ; subtract cd ; divide by d ; where bd and cd are announced as the answer to multiplying any number b by d (of course $d \neq 0$).

Task 3.4.4 Undoing Operations

Given a sequence of arithmetic operations such as the one used earlier (add 5, multiply by 3, subtract 1, divide by 2), find a starting number which produces 13 as the answer.

More generally, work out a method to find a starting number which will produce a specified result. Characterise the possible integer answers which will arise from an integer starting point.

Suggestion

Denote the answer by t . The operation sequence reduces to $t = 3s/2 + 7$, which when undone, becomes $x = (t - 7) \times 2/3$. So t must be 7 more than a multiple of 3, or, more simply, 1 more than a multiple of 3, in order that x be an integer.

Task 5.1.1b Compounded Percentages

Suppose that you add 10% of something and then add 10% of that. What is the overall percentage increase? Generalise.

Suggestion

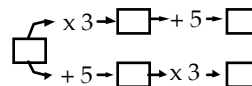
Let a and b be the two percentages being applied. Then the calculation is $(1 + b/100)(1 + a/100)$ which, since multiplication is commutative, is the same as $(1 + a/100)(1 + b/100)$. Both are equal to

$1 + (a + b + ab/100)/100$, so the overall percentage increase is the sum of the percentages, plus their product divided by 100. This suggests an 'arithmetic of percentages' in which

$$a \oplus b = a + b + ab/100.$$

Task 5.1.2a Different Routes

Write down any number. To the right of it and above, multiply by 3 and add 5. To the right and below, add 5 and then multiply by 3. Your two answers differ by 10. Why?



Suggestion

Suppose the audience secretly select an input, but are explicit about one of the outputs (and which one it is!). Then you can predict the other output without knowing the input they used, by using the fact that the difference is always 10.

Task 5.1.2b Different Routes Developed

What dimensions of possible variation could be exploited to generate a whole domain of tasks similar to this one with the same aim, namely to alert learners to the fact that operations performed in different orders give different answers, but there may be a relation between those answers?

Suggestion

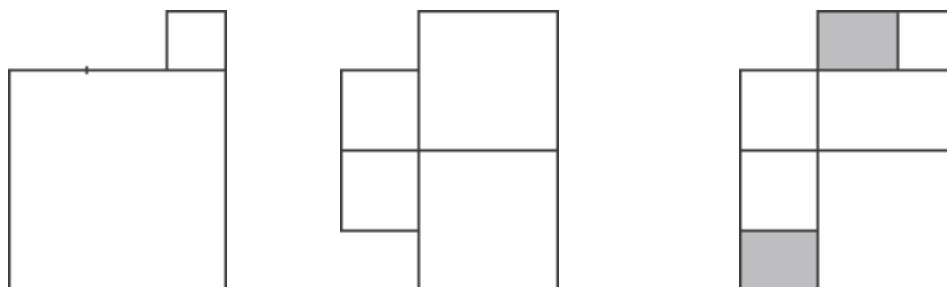
Begin by generalising the 3 and the 5: use m to denote the multiplier and use a to denote the adder. If the input number is x (unknown), then the two outputs would be $mx + a$ and $m(x + a)$ which differ by $m(a - 1)$. This can be found either by generalising from particular instances, or by manipulation of symbols.

Task 5.2.1 Interpreting Euclid

If a straight line is bisected, and a straight line is added to it in a straight line, then the square on the whole with the added straight line and the square on the added straight line both together are double the sum of the square on the half and the square described on the straight line made up of the half and the added straight line as on one straight line.

Suggestion

Here is a sequence of diagrams being built up according to the statement, with the third being a superposition of the first two. It might help to insert symbols for edge lengths (something Thales and his colleagues would never have done!).



The third diagram can even be read as a justification: the shaded rectangle at bottom left fits perfectly to complete the upper square, showing the two areas are the same.

Task 5.2.4b More Same and Different Multiplications

What is the same and what different about the multiplications in Task 5.2.5a and the short form below? Why does 'it' always work, with your interpretation, and what precisely is the 'it'?

Suggestion

Put the powers of x into the table and then follow the multiplying and the adding.

	$7x$	5	
	$7x^3$	$5x^2$	$1x^2$
$7x^3$	$14x^2$	$10x$	$2x$
$19x^2$	$21x$	15	3
$31x$	15		

Task 5.3.1a Reduced For Quick Sale

A supermarket advertised packets of mince tarts at 80p per packet with a second packet half-price. As the sell-by date approached, a further reduction was made to 40p each while retaining the 'second packet for half-price' sign. What then do you expect to pay for two packets? Express a general formula for this calculation.

Suggestion

Let p be the original item price in pence, say. Note that the p stands for price, not pence! Let r be the reduction ratio for the second item, so the price for two is $p + rp$. Let the revised price be q near the sell-by date. Then the store charges $q + rq - (1 - r)p$ for two when it is close to the sell-by date. This would be negative (!) if $q / p < (1 - r) / (1 + r)$. For $r = 1/2$, this means when the reduction of p to q is by more than $2/3$.

For example, if the two fruit are posted at 50p each or second one half-price, then reduced to 10p each to clear, the supermarket formula will charge the customer $10 + 5 - 25 = -10p!$

Quickie 5.3

Write down an expression for the angle between the minute hand and the hour hand of an analogue clock, in terms of the 'time' each shows.

Suggestion

Using the expressions you developed in the previous two sections, the angle between the hands is the difference between those angles. But the difference can be taken in either order, so 'the' angle has to be thought about carefully.

Task 5.4.3a Table Patterns

What is the range of permissible change encompassed by the 'this always' in the addition relation? Does it apply equally to the multiplicative relation?

Suggestion

The multiplicative relation exposes an invariant for each table, but that invariant depends on the values used in place of 1 and 7 (but not the 3). The additive invariant is independent of choice of numbers in place of 1, 7, and 3.

Quickie 5.5

Using your experience of the previous quickies in this chapter, write down an expression for the angle between the minute hand and the hour hand of an analogue clock in terms of the time the two hands 'show'.

Not so quick: how many times in a twelve hour period is the angle between the hands 0° ? 90° ?

Suggestion

Expressing the angle between the hour hand and the 12 o'clock position is expressing a generality.

Let h be the hour (notice that it is an integer, $1, \dots, 12$).

Then $(h/12)$ is the proportion of a full revolution of 360 degrees, so $(h/12) \times 360 = 30h$ is the angle (up to multiples of 360) of the hour hand when at h o'clock, with respect to the 12 position.

Similarly, the minute hand at m minutes past the hour (m is a number greater than or equal to 0 and less than 60) makes an angle of $(m/60) \times 360 = 6m$ with the 12 position.

But at m minutes past the hour, the hour hand has progressed by a further $(m/60)$ of $(360/12)$ degrees, or $m/2$. So the hour hand at m minutes past the hour makes an angle of $30h + m/2$ with the 12 position.

Task 6.1.1 Repetitions (again)

Task 5.1.2 (Repetitions), presented an operation (double and subtract one), to be repeated a specified number of time. Can any pre-assigned whole number be reached by applying this operation to some whole number? By applying it twice? Thrice? ... ? If so, how do you choose the starting number; if not, which ones can be reached?

Suggestion

Applying the operation once, doubles (making it even, then subtracts one, forcing the result to be odd. Any odd number can be reached in this way, since starting with 1 gives 1, starting with 2 gives 3, and so on. Applying the operation twice, you can get any number which leaves a remainder of 1 on dividing by 4. This can be generalised. The more often the operation is carried out, the fewer numbers can be reached.

Task 6.1.2a Number Forms

What numbers arise by multiplying two numbers which differ by 2 then adding 1?

What numbers arise by multiplying two numbers which differ by 4 and adding 4?

What numbers arise by multiplying two numbers which differ by 6 and adding 9?

Generalise!

Suggestion

For the first question, by trying particular cases, you might recognise the answers. Can you relate the answers to the original two numbers? Experience on the first question might suggest conjectures for the second and third questions, which when expressed generally, produces the subsuming generalisation.

Task 6.1.3 Characterising

What positive whole numbers have an odd number of factors?

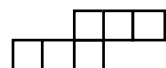
What straight-line graphs have their slope equal to their y -intercept? Half their y -intercept? More generally, t times their y -intercept?

What pairs of straight lines through the origin have their slopes the reciprocal of each other?

Suggestion

After trying some particular cases (specialising) you might have been led to think structurally: factors of a number come in pairs (the two factors whose product is the number), so for an odd number of factors, there must be a repeated pair, which means a square-root! Thus a number with an odd number of factors must be a perfect square. But do all square numbers have an odd number of factors? Being a square number characterises the numbers with an odd number of factors, and vice versa.

Task 6.3.1 Grid Lock



The shape shown has an area of 6 units and a perimeter of 14. How many other shapes can you find with the same area and the same perimeter, made up of the same squares?

Can you make a shape with the same area but smaller perimeter? Try a similar task on triangular paper.

Suggestion

For s identical squares, the single strip has an area of $2s + 2$. The more squares which share a common edge, the smaller the perimeter. Indeed, if c squares share a common edge, then the perimeter will be $4s - 2c$. This is achieved by making the largest square possible, then stacking more rows of that size until they

are all used. There will be a rectangle of $\lfloor \sqrt{s} \rfloor$ by $\left\lceil \frac{s}{\lfloor \sqrt{s} \rfloor} \right\rceil$ with the rest in a final short row on top, if

need be. So when $s = \lfloor \sqrt{s} \rfloor \times \left\lceil \frac{s}{\lfloor \sqrt{s} \rfloor} \right\rceil$ there will be no need for an extra row, but if $s > \lfloor \sqrt{s} \rfloor \times$

$\left\lceil \frac{s}{\lfloor \sqrt{s} \rfloor} \right\rceil$ then there will be one extra row. This gives a perimeter of $2 \left(\lfloor \sqrt{s} \rfloor + \left\lceil \frac{s}{\lfloor \sqrt{s} \rfloor} \right\rceil \right)$ when no extra

row is required, and an extra 2 units when the extra row is required. For example, for $s = 6$,

$\lfloor \sqrt{6} \rfloor \times \frac{6}{\lfloor \sqrt{6} \rfloor} = 6$ so the minimum perimeter is $2(2 + 3) = 10$. For 7, $\lfloor \sqrt{7} \rfloor \times \left\lceil \frac{7}{\lfloor \sqrt{7} \rfloor} \right\rceil = 6$ so the

minimum perimeter is $2(2 + 3) + 2 = 12$.

Task 6.4.5 Areas and Sequences

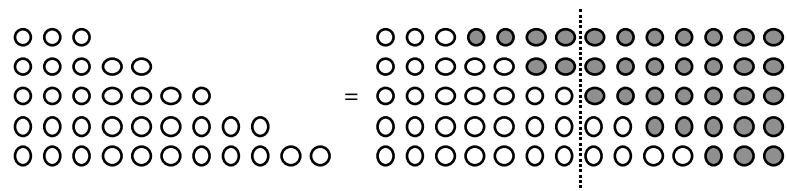
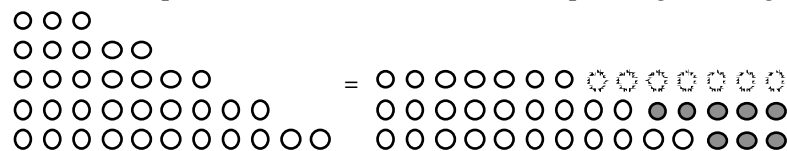
For each of the two rearrangements Find ways to depict the sum of an AP which leads directly to

$$\frac{n}{2}(a+b) \quad \text{and} \quad n\left(\frac{a+b}{2}\right)$$

as expressions of generality.

Suggestion

Here are two re-arrangements. Match them with the two formulae, then draw the equivalent diagram for areas of a trapezium and write down the corresponding rearrangements of the formula for its area.



Note that when n is even, the first diagram arrangement would not need the dashed circles, which are only shown here so as to explain the half.

Task 7.1.1b Hexagons Again

Translate each of the following shadings of the 5th hexagon in a sequence of larger and larger hexagons into a way of expressing the number of circles needed to make the n^{th} hexagon. Check that they do actually work on another case!

1: $1 + 6(n - 1) + 6(n - 2)(n - 1)/2$

2: $1 + 6n(n - 1)/2$

3: not at all obvious! Shading doesn't actually generalise!

4: $(n - 1)(n + 1) + n^2 + 2(n - 2)(n - 1)/2$

5: $H_{n+1} = H_n + 4(n - 1) + 1$

6: $n^2 + (n - 1)^2 + 2(n - 1)(n)/2$

7: $2n(n + 1)/2 + n(n - 2) + (n - 1)^2$

8: $(n - 1)(n)/2 + n^2 + (n - 1)^2$

Task 7.2.1b Closure Extended

None of the following collections of objects are closed under the indicated operations. What larger classes of objects would you need to use in order to achieve closure under the specified operation?

Squares under being cut in half (according to area) by a single straight line parallel to one edge of the square;

Triangles being cut into two pieces with equal perimeter by a single straight line parallel to one edge of the triangle;

Sectors of circles being cut into two pieces by a single straight line through the centre of the circular arc;

Whole numbers under subtraction;

Whole numbers under division by non-zero whole numbers;

Positive fractions under square-rooting.

Suggestion

Cutting a square in half by area parallel to an edge yields two rectangles with one side half the other. Cutting this rectangle yields a square or a rectangle with sides in the ratio of 4 : 1. repeated cutting produces rectangles with edges in the ratio of $2n : 1$ for $n = 0, 1, 2, \dots$. So the collection of rectangles with these proportions is closed under this operation.

Cutting a triangle in two pieces with equal perimeter yields a similar triangle to the original, and a trapezium. Continued cutting parallel to an edge yields a rhombus with sides in the ratio of 7:4 and a trapezium, or two trapezia. The collection of all rhombuses and symmetric trapezia is closed under the operation.

Task 7.2.6 Logarithms

Using the definition, interpret the following statements in terms of logarithms:

$3^2 = 9$; $2^3 = 8$; $10^3 = 1000$;

What base is implied by the statement $10^{\log(100)} = 100$?

Make up your own examples, both simple and complicated.

Decide whether the following expressions are always, sometimes or never true.

$\log_b(b) = 1$ $\log_b(b^n) = n$.

In what sense are powers of b and logarithms to base b inverse operations?

Decide whether the following expressions are always, sometimes or never true.

$$\log_b(a^n) = n\log_b(a) \quad \log_b(n \times m) = \log_b(n) + \log_b(m) \quad \log_b(n) = \log_a(n)/\log_a(b)$$

Suggestion

$\log_3(9) = 2$; $\log_2(8) = 3$, $\log_{10}(100) = 3$. The base implied is base 10 since the base raised to the log of a number to that base is the original number.

All the statements are always true as long as b is a positive number.

Task 9.1.3 LCM & GCD

What relationships can you find between the GCD and the LCM of two numbers?

Suggestion

Thinking in terms of prime factors, the HCF-GCD records all the factors in common, while the LCM records the largest number of times each prime appears in at least one of the numbers. Together, the HCF-GCD and the LCM record the total number of times each prime occurs in either of the numbers (e.g. the GCD of p^2 and p^3 is p^2 and the LCM is p^3). So the product of the HCF-GCD and the LCM must be the product of the two numbers. If you know the numbers and the HCF-GCD then you know the LCM, and vice versa.

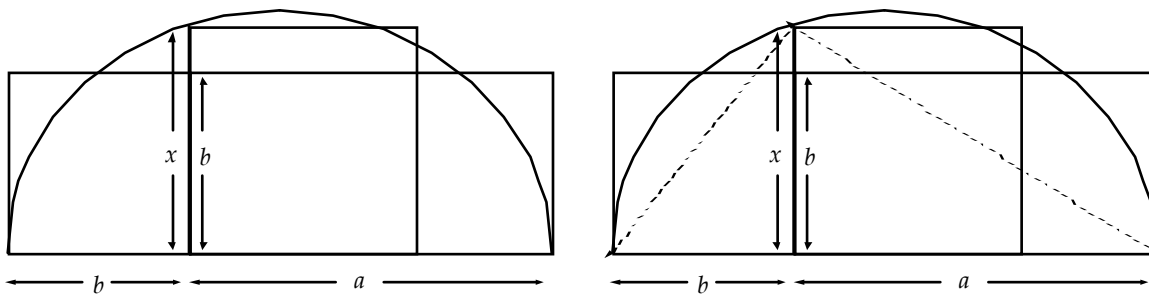
Task 9.2.1c From Rectangle to Square

Tell yourself a story which connects the pictures above into a story for how to convert a rectangle into a square of equal area. You will probably need to use Pythagoras' theorem on three different triangles which you will need to create for yourself!

Suggestions

On the final drawing insert some measurements which reflect the construction (add the vertical side of the rectangle to the horizontal side, draw a semicircle on that combined segment, use the segment from the division point of the extended side up to the circle as the edge of a square.

To prove it works, insert the dotted lines and observe three right-angled triangles. Expressing Pythagoras' theorem for each triangle and combining leads to $x^2 = ab$ as required.



Task 9.3.2 Bundling Up

If a string around a bundle of uniform sized sticks measures s cm, not including any knots etc., how many more sticks will there be when the string is twice as long? Put another way, if a spaghetti measurer for a single portion involves a hole with a diameter of 2.5cm, what diameter hole will measure twice as much spaghetti? If asparagus sells at 80p a bunch, what should be charged for a bunch twice the diameter?

Suggestion

For two similar figures (one a scaled up version of the other), doubling the perimeter scales the area by 4 (think of two squares, one with twice the side length of the other). Consequently, to double the area, you multiply the perimeter by $\sqrt{2}$, or roughly make a 41% increase (since $\sqrt{2} = 1.414\dots$).

Task 9.3.4 Exclusion

It was reported that during 1992 in a certain county, although black students were only 10% of the school population, 40% of school exclusions were black. This means, it said in the report, that black students are 6 times as likely to be excluded as white. Justify, comment and construct a general method for making similar calculations from other similar data.

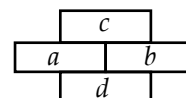
Suggestion

First, what is being compared in the final assertion? It presumably means the ratio of (the ratio of excluded black to black students) to (the ratio of excluded non-black to non-excluded non-black). Using a rectangle to represent all students, one division of the rectangle represents black and non-black students; another division represents excluded and non-excluded students. Let E be the number of students excluded, and S the number of students over all. Alternatively you might want to take s as some convenient number such as 100. The data can be used to work out the final ratio, perhaps by assuming that there are 100 students in the county.

There are of course all sorts of questions to ask about modelling assumptions that have been made.

Task 9.4.1a Reconstructing

Whenever two numbers appear in the middle row of the diagram, the sum is entered into the top box and the difference (left subtract right) goes in the bottom box.



Suppose that two of the entries have been erased. Is it possible to reconstruct the other entries?

Suggestion

Try inserting different letters in all the boxes, then writing down all the relationships. Try to find relationships which express each letter in terms of just two of the others. For example $c = a + b$, but also $d + b = a$. Once each letter is expressed in terms of two others, it is possible to pick any pair and then express the other two in terms of them. Some seventeenth century algebra texts provided tables of solutions for these and similar problems, inspired by the first known collection of algebra problems by Diophantus in about 250 CE. His first problem was, given the sum and difference of two numbers, to find the numbers, which he then demonstrates how to do using 10 as the sum and XXX as the difference. His problems get much harder, very quickly!

Task 9.4.1b Dimensions of Possible Variation

What other pairs of operations could be used in place of sum and difference?

Suggestion

Trying product and quotient, and denoting the top box by p and the bottom by q leads to the relationships $qb = a$, $p/a = b$, $q/a = b$ and so on. Expressing these all in terms of a and b reveals relationships such as that from $q = a/b$ you deduce that $qb = a$. Note the parallels with adding and subtracting. By reflecting on the relationships which emerge, you can develop rules for rearranging equations.

Task 9.4.1c Reconstructing Challenge

The operations this time are LCM (above) and HCF-GCD (below). Can you reconstruct the middle numbers given the top and bottom numbers? Are there any constraints on the pairs of numbers which can be used for the top and the bottom in order to be able to reconstruct the middle pair?

Suggestion

The HCF-GCD tells you the prime powers which have to be in both the numbers being sought. Divide the top number by the bottom. This number when factored tells you the prime powers which need to appear in just one of the two numbers. Then factor the quotient as ab such that a and b have no common factors. The middle numbers are then the bottom number times a and times b . Why?

Task 9.4.3 Tabled

Here is a 2 by 2 table with row and column sums. What is the maximum number of entries which could be rubbed out and still the full table could be reconstructed? Find a way to describe all the sets of cells whose contents could be rubbed out, and still the table reconstructed.

Generalise to r rows and c columns plus the row and column of sums.

Suggestion

Filling in the cells without the sums requires rc entries, so there are $r + c + 1$ entries that could be rubbed out. If any further entries are rubbed out, the table entries cannot be reconstructed because the final rubbed out entry becomes a dimension of possible variation. But can you rub out *any* $r + c + 1$ entries? If two entire rows or columns are rubbed out, then certainly it cannot be reconstructed; if four cells forming a two by two array (not necessarily adjacent) are rubbed out, then they cannot be reconstructed. Consequently, you can rub out up to $r + c + 1$ entries as long as there is no

If you are given redundant information (such as an entire row and its sum) then one further entry could be rubbed out. If every row (there are $r + 1$ rows) has one entry rubbed out, then the rest can be reconstructed, but there is redundant information in the columns, so an entire column's worth (c columns) can also be rubbed out.

Task 10.4.1 Sums and Products

What is the same and what is different about the structure in the following story problems (taken from Heath's translation of Diophantus who wrote around 250 CE)?

To divide a given number into two, having a given difference (Heath p130).

To divide a given number into two, having a given ratio (Heath p130).

To find two numbers in a given ratio such that their difference is also given (Heath p131).

Notice that the word 'divide' here means to split a number into the sum of two parts!

Suggestion

The first is Diophantus' version of a 'sum and difference' problem. For the second, let the given number be N , and let the given ratio be r . If one number is x , the other is rx , and since $x + rx = N$, $x = N/(1 + r)$. Diophantus' solution is entirely in words.

Quickie 11.2

Is it true that between any two distinct rational numbers there is an irrational number?

Suggestion

Denote the two rational numbers in decimal notation by $d.d_1d_2d_3\dots$ and by $D.D_1D_2D_3\dots$ where each d_k and each D_k is a decimal digit. (the property of being rational has not yet been expressed, because it is not needed). Then somewhere the d s and the D s must start to differ. Can you now insert a new decimal number in between? Since the property of a rational number is that it has a repeating decimal tail, can you arrange for that inserted number not to have a repeating tail?

Task 11.2.4 Putting It All Together

Express a relationship between $\binom{n}{r}$, $\binom{n-1}{r}$ and $\binom{n-1}{r-1}$ which also expresses a pattern you discerned in the number triangle in Task 9.1.1 and in Task 9.1.2.

Using the direct means of calculating $\binom{n}{r}$, show that your expressed pattern is actually correct.

Suggestion

It might help to gain experience in the use of the factorial notation to go through a particular case.

In general, since $\binom{n}{r} = \frac{n(n-1)(n-2)\dots(n-r+1)}{r(r-1)(r-2)\dots 1}$, $\binom{n-1}{r} = \frac{(n-1)(n-2)\dots((n-1)-r+1)}{r(r-1)(r-2)\dots 1}$ and

$$\binom{n-1}{r-1} = \frac{(n-1)(n-2)\dots((n-1)-(r-1)+1)}{(r-1)(r-2)\dots 1}$$
 adding the last two gives

$$\frac{(n-1)(n-2)\dots((n-1)-(r-1)+1)}{(r-1)(r-2)\dots 1} + \frac{(n-1)(n-2)\dots((n-1)-r+1)}{r(r-1)(r-2)\dots 1}$$

Notice that the denominators are the same except for an extra r in the second, and the numerators are the same except for an extra $n-r$ in the second. So taking out common factors gives

$$= \frac{(n-1)(n-2)\dots(n-r+1)}{(r-1)(r-2)\dots 1} \left(1 + \frac{n-r}{r}\right)$$

$$= \frac{(n-1)(n-2)\dots(n-r+1)}{(r-1)(r-2)\dots 1} \left(\frac{n}{r}\right)$$

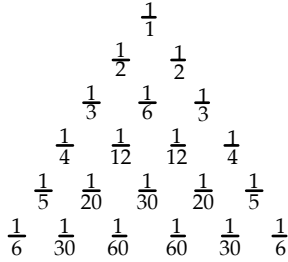
$$= \frac{n(n-1)(n-2)\dots(n-r+1)}{r(r-1)(r-2)\dots 1}$$

$$= \binom{n}{r} \text{ as expected!}$$

What is important is developing confidence to work with letters and to let them support the general reasoning.

Task 11.2.5 Leibniz's Triangle

Look for some relationships among entries in this triangular array of numbers. Find a link with the Jia Xian or Pascal triangle as well.



Suggestion

Notice the symmetry, and the first and last entries in each row.

Look at the denominators in each row and relate these to the rows of the Jia Xian triangle.

Notice that the sum of two entries side by side is the entry above them (how does this relate to the Jia Xian triangle relationship?)

By specifying an entry in this triangle using entries in the other triangle, can you then verify that the sum of two adjacent entries in a row is the entry above them?

Leibniz's triangle is very useful for summing infinite series! For example, the top $1/1$ is the sum of the two $1/2$ s in the second row; the right hand of these is the sum of the $1/6$ and the right hand $1/3$; the $1/3$ is the sum of the $1/12$ and the $1/4$; and so on, suggesting that $1/1$ is the sum of $1/2$ and $1/2$ which is the sum of $1/2$ and $1/6$ and $1/3$ which is the sum of $1/2$ and $1/6$ and $1/12$ and $1/4$ which is the sum of So perhaps

$$1 = 1/2 + 1/6 + 1/12 + 1/20 + 1/30 + \dots$$

Similarly, the left hand $1/2$ is the sum of $1/2$ and $1/6$ which is the sum of $1/3$ and $1/12$ and $1/12$, which is the sum of $1/3$ and $1/12$ and $1/30$ and

These sums are only suggestive: advanced techniques are required for verifying that the infinite sums actually make sense.

To verify that the Jia Xian triangle used to generate the Leibniz triangle gives the correct rule for combining terms, proceed as follows.

The denominator of the r th entry in the n th row of Leibniz's triangle is $n \binom{n-1}{r}$ for r from 0 to $n-1$.

Task 11.3.2b Rearranging Inequalities

Starting from the true statement that $2 < 3$, perform operations on both sides of the inequality in order to show that the following are also true. Generalise each operation (but take care!)

$$2 \times 5 < 3 \times 5 \quad 2 > 3 \quad 2 + 4 < 3 + 4 \quad 2 - 5 < 3 - 5$$

Now perform the same operations on the inequality $a < b$ to obtain rules for manipulating inequalities.

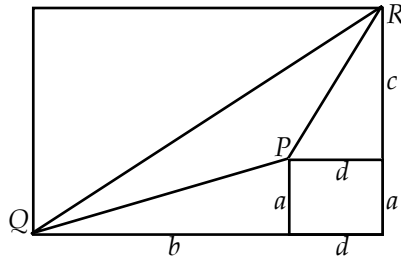
Suggestion

Adding (and hence subtracting) the same number top (from) both sides of an inequality preserves the inequality.

Multiplying (and hence dividing) both sides of an inequality by the same *positive* number preserves the inequality. Multiplying or dividing by a negative number reverses the inequality (think of a number line with two points on it; reflect the number-line in the point 0, that is, multiply each entry by -1 ; the order of the two points is reversed by this action).

Task 11.3.4 Reading Inequalities

Express a relationship between the slopes of the three diagonal lines.



What constraints are there on the position of the point P in order to preserve your inequality?

Suggestion

The slope of PQ is less than the slope of QR , while the slope of PR is greater than the slope of QR and QP . The slope of PQ is a/b and the slope of QR is $(a + c)/(b + d)$, while the slope of PR is c/d . Thus $\frac{a}{b} < \frac{a + c}{b + d} < \frac{c}{d}$. These inequalities remain true as long as P lies below the diagonal within the rectangle.

Actually, it can stray even farther, but not everywhere below the diagonal line! This range of permissible change for P means that the inequality has to be qualified:

$$\text{if } \frac{a}{b} < \frac{c}{d} \text{ then } \frac{a}{b} < \frac{a + c}{b + d} < \frac{c}{d}$$

or alternatively, that the fraction formed by adding the numerators and the denominators of two fractions lies between their values, as long as negative slopes have the negative sign attached to the numerator.
